A survey of gravitational waves, the Bondi-Sachs formalism and the gravitational memory effect in general relativity and beyond

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In loving memory of Guillermina del Carmen Obreque Vega

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Abstract

This thesis is an introductory recollection of the results from the original work of Bondi, Van der Burg, Metzner and Sachs in the context of gravitational waves in general relativity.

It starts with a brief summary of the linearized weak-field theory and argues its possible shortcomings when describing non-linear effects. It continues with the construction of an axially symmetric, non-stationary metric based on a family of null hypersurfaces enumerated by a null-time coordinate. The solutions to the vacuum field equations of this metric, presented originally by Bondi, lead to the news function and the mass-loss and momentum-loss formulae. These results explain the change of mass and momentum of a system through the emission of gravitational waves as a consequence of the non-linear nature of the gravitational field equations.

The final part of this thesis presents the Bondi-Sachs metric as a generalization of the Bondi metric based on the same family of null hypersurfaces. This new metric simplifies the introduction of more abstract concepts like asymptotic symmetries and the algebras induced by them. Previous results are thus translated into this new form and the advantages of the Bondi-Sachs metric are used to briefly cover the example of the memory effect, a permanent relative displacement due to the propagation of gravitational waves. This is followed by a discussion on the possibility of using the loss formulae together with the memory effect as a method to discard, validate or distinguish alternative theories of gravity.

Contents

Introduction

In his work from 1916, Albert Einstein found the linearized weak-field solutions to his equations. One of the consequences of these solutions was the prediction of gravitational waves [6]. Though their experimental confirmation would not be achieved until hundred years later [1], there was in the 1960s the open theoretical question, whether the non-linear nature of gravitational field equations could lead to effects not predicted by the linear theory. The original work from 1962 by Bondi, Van der Burg, Metzner and Sachs attempts to answer this question by considering an asymptotically flat metric at null infinity [4].

One surprising result of the work from 1962, was the discovery of the BMS group. In flat spacetime, the metric allows for a ten dimensional group of isometries that constitutes the Poincaré group. For the asymptotically flat case, however, to this group, an additional, infinite, family of transformations, called super-translations, is incorporated.

The focus is this thesis is on a further result of the work by Bondi, Van der Burg, Metzner, and Sachs. Specifically, the description of the dynamics of the system, and in particular, the change in mass and momentum as a consequence of the emission of gravitational waves. This is described by the news function and the null-time derivatives of the mass and momentum aspects.

Although primarily written in the context of gravitational waves, the results of the 1962 work, have recently received much interest in modern times in the search for a theory of quantum gravity, where there exists proposals to gain intuition through symmetries of the BMS group and the algebras they induce [5]. Fundamental in this modern context is the generalization of the metric introduced by Sachs in [12].

The goal of this thesis is to provide a detailed introduction and overview of the initial results of the Bondi-Sachs formalism. Follow up work shall study the dynamics of gravitational waves in the Bondi-Sachs formalism in theories of gravity beyond general relativity, and identify the difference, for example in the mass loss formula and the memory effect.

Structure of this thesis

Chapter 1 starts with a very brief discussion on the linearized weak-field solutions of the Einstein vacuum equations and the mathematical existence of gravitational waves in this limit of linearized gravity. Chapter 2 continues with the construction of the Bondi metric, used for the description of the more general, non-linear, case, and the resulting gravitational field equations. This part of the thesis, following the ideas first presented by Bondi and Metzner in their paper of 1962 [4], attempts to complement them by giving easier access to their results. Chapter 3 treats a generalization of the Bondi metric, introduced by Sachs, which disposes of the restriction to consider axially symmetric systems. A translation of the main results of Chapter 2 into the Bondi-Sachs metric is provided together with a brief mention of the memory effect. Finally Chapter 4 closes by discussing the idea of an extension of the Bondi-Sachs formalism to alternative theories of gravitation.

(CAS) Maxima code

One of the objectives of this work is, not only to explore the works of already existing literature, but to corroborate and reproduce their analytic results, opening the possibility to repeat the process in the future for new proposals for gravitation theories. Therefore, much work was done to write a set of maxima scripts that calculate the components of the Christoffel symbols, and Ricci tensor for the different metrics considered, as far as this was possible. The source code is provided together with the digital version of this text and also available online at `https://git.thomaslabs.org/bondi-sachs/`.

Notation and conventions

In this paper, when dealing with results valid independently of the chosen coordinate system, the abstract index notation as described by Wald [17] is used, i.e. lowercase letter included as needed starting from a. Otherwise the tensor quantities are to be indexed by lowercase greek letters enumerating components from 0 up to 3. An exception is made in Chapter 3. Here we introduce uppercase latin letters to denote the components of the round 2-sphere metric created by the two angular components, such that $A, B, C, \dots = 2, 3$.

As for the derivatives of tensor quantities, these will be expressed as ∇_a, or any other lowercase latin letter, for the Levi-Civita connection and ∂_μ, with μ possible replaced for a number between 0 and 3, for the family of ordinary partial derivatives.

Partial derivatives, where convenient, are also represented by the variable being differentiated by as a sub-index, e.g. $f_u, f_r, f_\theta, f_\phi$. Special care was taken, as to keep this notation unambiguous with respect to the tensor indices, i.e. a coordinate as a sub-index is to be interpreted exclusively as a partial derivative.

Throughout this work we make use, without exception, of the $(-,+,+,+)$ signature. Some results taken from literature written with a different signature may appear different as a consequence of this.

All results are expressed using geometrized units, i.e.

$$G = c = 1$$

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Chapter 1

A brief overview of linearized gravity

The Bondi-Sachs formalism, as later introduced in Chapter 2, and further developed in Chapter 3, emerges as an attempt to answer open questions left by the linearized weak-field equations, and their solutions in form of gravitational waves, first found by Einstein in 1916 [6].

As to illustrate the need for this formalism we dedicate this chapter to feature the simpler case of linear gravity, following the steps presented by Wald [17]. Here, we define the metric of the spacetime, g_{ab}, as a perturbation of the Minkowski metric η_{ab} by

$$g_{ab} = \eta_{ab} + \epsilon\gamma_{ab}, \qquad g^{ab} = \eta^{ab} - \epsilon\gamma^{ab}, \tag{1.1}$$

where ϵ is an infinitesimal such that results are considered only up to the first order. Additionally, for lowering and raising indices we are to use η_{ab} and η^{ab}, with the exception of the metric itself being g^{ab}.

Using this simplifications, in addition to the gauge freedom

$$\gamma_{ab} \to \gamma_{ab} + \partial_a\xi_b + \partial_b\xi_a, \tag{1.2}$$

where

$$\partial^b\partial_b\xi^a = 0, \tag{1.3}$$

the Einstein vacuum equations $R_{ab} = 0$ takes the form

$$\partial^a\gamma_{ab} = 0, \tag{1.4}$$

$$\partial^c\partial_c\gamma_{ab} = 0. \tag{1.5}$$

With the exception of the extra gauge freedom, the linearized Einstein equations look formally similar to the vacuum Maxwell equations.

Though details were left here under the assumption of this being a known result in the general literature, the important conclusion becomes clear. Spacetime itself can, at least in the linearized case, propagate oscillations of the metric, which under this approximation, behave similar to electromagnetic waves.

However, due to the non-linear nature of Einstein equations, there is a priori, no reason to believe that a description applies also in the general case. Therefore, it is necessary to look for a more general approach to describe the propagation and effects of gravitational waves.

Chapter 2

The Bondi metric

In Chapter 1 we argued for the theoretical existence of gravitational waves in the context of the Einstein's theory. However, being restricted to the use of a linearized theory leaves questions open, as to the possible phenomena due exclusively to the non-linear nature of the gravitational field. For example, the existence of tails due to back scatter and spreading behind the moving pulse [11, 4].

It is well known that the lowest kind of symmetry which one can associate with gravitational waves is that of a quadrupole. Conservation of mass restrict waves from being purely spherically symmetrical and conservation of momentum, similarly, prohibits waves of dipole symmetry. On the other hand, from the transmission of energy it becomes clear that there has to be a change in the mass of the system. To account for this let us consider a system initially and finally static and spherically symmetric but with a non-static non-spherically symmetric period sandwiched in between. We claim that higher terms react back through non-linearities and affect the mass and total momentum of the source.

This chapter aims to prove this statement. For this, we start in Section 2.1 by constructing the Bondi metric for describing a non-static, axially symmetric and asymptotically flat spacetime. Section 2.2 continues by presenting Einstein vacuum equation for the Bondi metric. As we shall see, the Bondi metric reduces the resulting number of equations to solve. In Section 2.3, we use a series of inverse powers to solve the equations derived in Section 2.2. A special coordinate transformation is introduced in Section 2.4 to further reduce the number of free parameters in our solution. Finally, Section 2.5 derives the most important result of this chapter, the mass-loss and momentum-loss formulae, which, as previously stated, explain the changes in mass and total momentum of the system.

2.1 Construction of the Bondi metric

As stated at the beginning of this chapter, we wish to describe a non-static, axially symmetric and asymptotically flat spacetime. Moreover, we describe our system from null infinity. We first introduce coordinates u, r, θ, ϕ being enumerated by 0, 1, 2, 3 and defined in such a way, that light rays propagate along the path generated by the variation of only the r component. Therefore, we have $g_{11} = 0$. We also require the azimuth angle ϕ to appear separately in the metric, so that $g_{03} = g_{13} = g_{23} = 0$. Thus, so far

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & 0 \\ g_{01} & 0 & g_{21} & 0 \\ g_{02} & g_{21} & g_{22} & 0 \\ 0 & 0 & 0 & g_{33} \end{pmatrix}. \tag{2.1}$$

The desired metric, as explained, should asymptotically tend towards the Minkowski metric, which due to the previously mentioned requirement and the coordinate transformation $u = t - r$ takes the form

$$g = -\mathrm{d}u^2 - 2\mathrm{d}u\,\mathrm{d}r + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \tag{2.2}$$

and has therefore the property of u being a time-like coordinate with $g_{00} < 0$.

Let us consider a vector k^a, tangent to a light ray. By definition, we have $g_{ab}k^a k^b = 0$ identically along the path of the ray and

$$k^c \nabla_c g_{ab} k^a k^b = 2k^c g_{ab}(\nabla_c k^a) k^b = 0. \tag{2.3}$$

By writing the right equation in our basis, this equation becomes

$$g_{\mu\nu}\left(\delta_1^\lambda \partial_\lambda \delta_1^\mu + \Gamma^\mu{}_{\lambda\rho}\delta_1^\rho \delta_1^\lambda\right)\delta_1^\nu = 0, \tag{2.4}$$

which clearly reduces to

$$\Gamma^0{}_{11}g_{01} + \Gamma^1{}_{11}g_{11} + \Gamma^2{}_{11}g_{21} + \Gamma^3{}_{11}g_{31} = 0. \tag{2.5}$$

Hence, from our condition for the azimuth angle and $g_{11} = 0$, it follows

$$\left.\begin{aligned} 0 = \Gamma^0{}_{11} = g^{00}\partial_1 g_{10} + g^{02}\,\partial_1 g_{12} \\ 0 = \Gamma^2{}_{11} = g^{02}\partial_1 g_{10} + g^{22}\,\partial_1 g_{12} \end{aligned}\right\}. \tag{2.6}$$

From equation (2.1) we know that neither $\det g$ nor g_{33} can vanish. Furthermore

$$\left.\begin{aligned} g^{00}\det g/g_{33} &= -(g_{12})^2 \\ g^{02}\det g/g_{33} &= g_{01}g_{12} \\ g^{22}\det g/g_{33} &= -(g_{01})^2 \end{aligned}\right\}. \tag{2.7}$$

Thus, equation (2.6) is equivalent to

$$\left.\begin{aligned} 0 = g_{12}(g_{12}\,\partial_1 g_{01} - g_{01}\,\partial_1 g_{12}) \\ 0 = g_{01}(g_{12}\,\partial_1 g_{01} - g_{01}\,\partial_1 g_{12}) \end{aligned}\right\}. \tag{2.8}$$

This implies that either $g_{01} = 0$, or that $g_{12} = 0$, or that $(g_{12}/g_{01})_r = 0$. Suppose now that $g_{01} = 0$. Then the determinant of equation (2.1), after performing a Laplace expansion along the second row of the minor, yields

$$\det g = -g_{33}(g_{12})^2 g_{00}. \tag{2.9}$$

From equation (2.2) we know that $g_{33} > 0$, hence for the signature of the metric to be correct it must hold that $g_{00} > 0$. However, u would no longer be a time-like coordinate, contrary to its definition. If $(g_{12}/g_{01})_r = 0$, then one can replace u by a variable $\bar{u}(u,\theta)$ such that g_{12} is reduced to zero [4]. Therefore the metric is characterized by the condition

$$g_{12} = 0 \tag{2.10}$$

As for the remaining components of the metric, we define r as to keep the area of the surface element with constant u and r to be $r^2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}r$.

Through this argumentation we arrive at the metric as first given by Bondi [3][1]

$$\begin{aligned} g = (U^2 r^2 \mathrm{e}^{2\gamma} - V r^{-1} \mathrm{e}^{2\beta}) \mathrm{d}u^2 \; - 2\mathrm{e}^{2\beta} \mathrm{d}u \, \mathrm{d}r \\ - 2U r^2 \mathrm{e}^{2\gamma} \mathrm{d}u \, \mathrm{d}\theta \; + r^2 (\mathrm{e}^{2\gamma} \mathrm{d}\theta^2 \; + \mathrm{e}^{-2\gamma} \sin^2\theta \, \mathrm{d}\phi^2 \,) \end{aligned} \tag{2.11}$$

With its inverse given by

$$g^{\mu\nu} = \begin{pmatrix} 0 & -\mathrm{e}^{-2\beta} & 0 & 0 \\ -\mathrm{e}^{-2\beta} & V\mathrm{e}^{-2\beta} r^{-1} & -U\mathrm{e}^{-2\beta} & 0 \\ 0 & -U\mathrm{e}^{-2\beta} & \mathrm{e}^{-2\gamma} r^{-2} & 0 \\ 0 & 0 & 0 & \mathrm{e}^{2\gamma} r^{-2} \sin^{-2}\theta \end{pmatrix} \tag{2.12}$$

with the functions γ, β, U, V being, due to the axial symmetry of the system, only dependent on u, r and θ.

From $g^{\mu\nu}$ it is evident, that the constructed metric describes a family of outgoing null hypersurfaces labeled by $u = t - r$, where the coordinate r is an areal coordinate with null rays enumerated by θ and ϕ [9].

2.2 The main equations

In the previous section we determined the form of the metric. In this section, we now consider Einstein's vacuum field equation

$$R_{\mu\nu} = 0, \tag{2.13}$$

where R is the Ricci curvature tensor. In general this comprises ten coupled, non-linear, differential equation. For the Bondi metric, however, it holds identically for the Ricci tensor

$$R_{03} = R_{13} = R_{23} = 0. \tag{2.14}$$

Thus, the system is simplified to only seven equations. Let us now furthermore suppose that

$$R_{11} = R_{12} = R_{22} = R_{33} = 0. \tag{2.15}$$

The Bianchi identities

$$g^{\mu\nu}(\partial_\nu R_{\lambda\mu} - \frac{1}{2}\partial_\lambda R_{\mu\nu} - \Gamma^{\delta}{}_{\mu\nu} R_{\lambda\delta}) = 0 \tag{2.16}$$

[1]Note the change from the original $(+,-,-,-)$ metric in the original work of Bondi. This was done for reasons of internal consistency of the present work.

are hence simplified into

$$\begin{aligned}0 &= g^{01}\partial_1 R_{00} + g^{11}\partial_1 R_{01} + g^{12}(\partial_2 R_{01} + \partial_1 R_{02}) + g^{22}\partial_2 R_{02} \\ &\quad - g^{\mu\nu}\Gamma^0{}_{\mu\nu} R_{00} - g^{\mu\nu}\Gamma^1{}_{\mu\nu} R_{01} - g^{\mu\nu}\Gamma^2{}_{\mu\nu} R_{02},\end{aligned} \tag{2.17}$$

$$0 = -g^{\mu\nu}\Gamma^0{}_{\mu\nu} R_{01}, \tag{2.18}$$

$$0 = g^{01}(\partial_1 R_{02} - \partial_2 R_{01}) - g^{\mu\nu}\Gamma^0{}_{\mu\nu} R_{02}. \tag{2.19}$$

With the help of the result

$$g^{\mu\nu}\Gamma^0{}_{\mu\nu} = -2r^{-1}\mathrm{e}^{-2\beta}, \tag{2.20}$$

it becomes evident, that equation (2.18) can only have the trivial solution $R_{01} = 0$. As a consequence of (2.20), the equations involving R_{00} and R_{02} are also reduced into the supplementary conditions

$$0 = r^{-2}\mathrm{e}^{-2\beta}(r^2 R_{02})_r, \tag{2.21}$$

$$0 = r^{-2}\mathrm{e}^{-2\beta}(r^2 R_{00})_r + (g^{12}\partial_1 + g^{22}\partial_2 - g^{\mu\nu}\Gamma^2{}_{\mu\nu})R_{02}. \tag{2.22}$$

In this manner, Einstein's vacuum equations are simplified to just the four equations (2.15), which from now on shall be referred by main equations, together with the conditions (2.21) and (2.22).

Finally we reformulate the main equations in a equivalent, though more practical form:

$$\begin{aligned}0 &= R_{11} \\ &= -4r^{-1}[\beta_r - \frac{1}{2}r\gamma_r^2],\end{aligned} \tag{2.23}$$

$$\begin{aligned}0 &= -2r^2 R_{12} \\ &= [r^4\mathrm{e}^{2(\gamma-\beta)}U_r]_r - 2r^2[\beta_{r\theta} - \gamma_{r\theta} + 2\gamma_r\gamma_\theta - 2r^{-1}\beta_\theta - 2\gamma_r\cot\theta],\end{aligned} \tag{2.24}$$

$$\begin{aligned}0 &= -\mathrm{e}^{2(\beta-\gamma)}R_{22} - r^2\mathrm{e}^{2\beta}R^3{}_3 \\ &= 2V_r + \frac{1}{2}r^4\mathrm{e}^{2(\gamma-\beta)}U_r^2 - r^2U_{r\theta} - 4rU_\theta - r^2U_r\cot\theta - 4rU\cot\theta \\ &\quad + 2\mathrm{e}^{2(\beta-\gamma)}[-1 - (3\gamma_\theta - \beta_\theta)\cot\theta - \gamma_{\theta\theta} + \beta_{\theta\theta} + \beta_\theta^2 \\ &\quad + 2\gamma_\theta(\gamma_\theta - \beta_\theta)],\end{aligned} \tag{2.25}$$

$$\begin{aligned}0 &= -r^2\mathrm{e}^{2\beta}R^3{}_3 \\ &= 2r(r\gamma)_{ur} + (1 - r\gamma_r)V_r - (r\gamma_{rr} + \gamma_r)V - r(1 - r\gamma_r)U_\theta \\ &\quad - r^2(\cot\theta - \gamma_2)U_r + r(2r\gamma_{r\theta} + 2\gamma_\theta + r\gamma_r\cot\theta - 3\cot\theta)U \\ &\quad + \mathrm{e}^{2(\beta-\gamma)}[-1 - (3\gamma_\theta - 2\beta_\theta)\cot\theta - \gamma_{\theta\theta} + 2\gamma_\theta(\gamma_\theta - \beta_\theta)].\end{aligned} \tag{2.26}$$

2.3 Series expansion of the coefficients

Before we continue with the solution of Einstein vacuum equation for the Bondi metric, we make a small detour and consider for a moment the Weyl metric given by

$$g = -\mathrm{e}^{2\psi}\mathrm{d}t^2 + \mathrm{e}^{2(\delta-\psi)}(\mathrm{d}\rho^2 + \mathrm{d}z^2) + \mathrm{e}^{-2\psi}\rho^2\mathrm{d}\varphi^2\,, \tag{2.27}$$

where δ and ψ are functions of ρ and z.

This metric describes any stationary and axially symmetric spacetime [17]. Furthermore, if we go back to the Bondi metric and consider intervals $a < u < b$, for which our system is stationary, it can be shown that there exists a transformation between the Weyl and the Bondi metric [4]. From the comparing the components of both metrics, an interpretation can be deduced for them. In particular, due to the condition of absence of inward flowing radiation the function γ has to take the form of a power series in the negative powers of r [4]. We take therefore the following ansatz for the variables

$$\left.\begin{aligned}\gamma &= \gamma_1 r^{-1} + \gamma_2 r^{-2} \dots \\ \beta &= \dots \beta_{-1} r + \beta_0 + \beta_1 r^{-1} + \beta_2 r^{-2} \dots \\ U &= \dots U_{-1} r + U_0 + U_1 r^{-1} + U_2 r^{-2} \dots \\ V &= \dots V_{-1} r + V_0 + V_1 r^{-1} + V_2 r^{-2} \dots\end{aligned}\right\}, \tag{2.28}$$

with each coefficient being a function of u and θ.

The results presented for $r \to \infty$ throughout this section are discussed in more detail in Appendix A.

Considering the structure of the main equations under ansatz (2.28), it is noted that given γ, β cannot include positive powers of r and is determined by equation (2.23) up to an integration function $\beta_0 = H(u, \theta)$. Moreover, $\beta_r = \mathcal{O}\left(r^{-3}\right)$ so that β remains bounded for $r \to \infty$. Plugging the expansion of γ into equation (2.23) also shows following relations between the coefficients

$$\beta_1 = 0, \qquad\qquad \beta_2 = -\frac{1}{4}\gamma_1^2.$$

For reasons explained later, γ_1 is deserving of a special name and shall from now on be referred as c.

Using γ and the from equation (2.23) derived result for β, equation (2.24) yields U up to an integration function $L(u, \theta)$ added directly to U and another $-6N(u, \theta)$ added to $r^4 \mathrm{e}^{2(\gamma-\beta)} U_r$. Additionally

$$U = L + \mathcal{O}\left(r^{-2}\right),$$

hence

$$\lim_{r \to \infty} U(r, u, \theta) = L(u, \theta). \tag{2.29}$$

Equation (2.25) then determines V, where an arbitrary integration function $-2M(u, \theta)$, called *mass aspect*, may also be added. Using the same reasoning for $r \to \infty$ and equation (2.29), we conclude that equation (2.25) takes the form

$$2V_r - 4rL_\theta - 4rL\cot\theta + K + \mathcal{O}\left(r^{-1}\right) = 0,$$

for some function $K(u, \theta)$. Therefore V has a leading term not greater than r^2. However it is now possible to see, that in order to keep the desired nature of the metric as written in equation (2.11), g_{00} has to remain bounded and hence $L = 0$ such that the leading of V is proportional to r.

Equation (2.26), being the only one involving differentiation with respect to u lastly determines γ up to a term $c_u(u, \theta) r^{-1}$. Therefore, given γ for a value of u and the four functions $H(u, \theta)$, $N(u, \theta)$, $M(u, \theta)$ and $c(u, \theta)$, the entire development is given by the main equations [4], with the components of the metric taking the form

$$\gamma = cr^{-1} + \dots \tag{2.30}$$

$$\beta = H - \frac{1}{4}c^2 r^{-2} + \dots \tag{2.31}$$

$$U = 2H_\theta \mathrm{e}^{2H} r^{-1} + \dots \tag{2.32}$$

$$V = r\mathrm{e}^{2H}[1 + 2H_\theta \cot\theta + 4H_\theta^2 + 2H_{\theta\theta}] + \dots. \tag{2.33}$$

2.4 Coordinate transformations of the metric

As showed in the previous section, one of the integration functions could be reduced to zero just by imposing the condition $g_{00} \to -1$. Similarly, it is also possible to reduce H to zero by performing a suitable coordinate transformation. With this in mind we introduce the family of coordinate transformations of the form

$$\left.\begin{aligned} u &= a(\bar{u},\bar{\theta})\bar{r} + a_0(\bar{u},\bar{\theta}) + a_1(\bar{u},\bar{\theta})\bar{r}^{-1} + \dots \\ r &= \rho(\bar{u},\bar{\theta})\bar{r} + \rho_0(\bar{u},\bar{\theta}) + \rho_1(\bar{u},\bar{\theta})\bar{r}^{-1} + \dots \\ \theta &= g(\bar{u},\bar{\theta})\bar{r} + g_0(\bar{u},\bar{\theta}) + g_1(\bar{u},\bar{\theta})\bar{r}^{-1} + \dots \end{aligned}\right\} \tag{2.34}$$

The coefficients are chosen as to preserve the character of the metric (2.11),

$$\bar{g}_{11} = 0, \qquad \bar{g}_{12} = 0, \qquad \bar{g}_{22}\bar{g}_{33} = \bar{r}^4 \sin^2\bar{\theta}$$

The details of such a family of transformations are deferred to the original work [4] and are discussed in a more general way by Sachs [12]. They give rise to the known BMS group of asymptotic symmetries of the metric.

As previously mentioned in the introduction, this group is one of the more unexpected results of the work of Bondi, Metzner and Sachs.It has itself received much recent attention when dealing with the asymptotic symmetries of spacetime, such as in the case for the search for a theory of quantum gravity [5, 7]. We additionally point the reader to the work by Bishop and Rezzolla [2].

Here, we present the coefficients in their final form:

$$\gamma = cr^{-1} + [C - \frac{1}{6}c^3]r^{-3} + \dots, \tag{2.35}$$

$$\beta = -\frac{1}{4}c^2 r^{-2} + \dots, \tag{2.36}$$

$$\begin{aligned} U = &-(c_\theta + 2c\cot\theta)r^{-2} + [2N + 3cc_\theta + 4c^2\cot\theta]r^{-3} \\ &+ \frac{1}{2}(3C_\theta + 6C\cot\theta - 6cN - 8c^2 c_\theta - 8c^3\cot\theta)r^{-4} + \dots, \end{aligned} \tag{2.37}$$

$$\begin{aligned} V = r - 2M &- [N_\theta + N\cot\theta - c_\theta^2 - 4cc_\theta - \frac{1}{2}c^2(1 + 8\cot^2\theta)]r^{-1} \\ &- \frac{1}{2}[C_{\theta\theta} + 3C_\theta\cot\theta - 2C + 6N(c_\theta + 2c\cot\theta) \\ &+ 8c(c_\theta^2 + 3cc_\theta + 2c^2\cot^2\theta)]r^{-2} + \dots, \end{aligned} \tag{2.38}$$

with C a function of u and θ such that

$$4C_u = 2c^2 c_\theta + 2cM + N\cot\theta - N_\theta.$$

With the final form of the coefficients of the metric, we proceed in the next section to derive the principal result of this chapter, the mass-loss and momentum-loss formulae.

2.5 The mass-loss and momentum-loss formulae

From equation (2.21) it is clear that R_{02} has to be of the form $f(u,\theta)r^{-2}$. Thus, if it is to vanish for some value of r and all values of u and θ, then it is to vanish for all values of r. Hence, as a consequence of equation (2.22), R_{00} also is of the form $f(u,\theta)r^{-2}$. Plugging the above expansion for γ, β, U and V into $R_{00}=0$ and $R_{02}=0$ and setting the coefficient of the r^{-2} term equal to zero yields respectively

$$M_u = -c_u^2 + \frac{1}{2}(c_{\theta\theta} + 3c_\theta \cot\theta - 2c)_u, \tag{2.39}$$

$$-3N_u = M_\theta + 3cc_{u\theta} + 4cc_u \cot\theta + c_u c_\theta. \tag{2.40}$$

From this result we see that the functions are fully determined if $c(u,\theta)$ is given and the value of M and N is known for some value of u.

As previously mentioned, the overlap of the Bondi metric with the Weyl metric for the static case allows to give meaning to the values of the parameters. This way we define the mass $m(u)$ of the system as the mean value of the mass aspect $M(u,\theta)$ over the sphere

$$m(u) = \frac{1}{2}\int_0^\pi M(u,\theta)\sin\theta\,\mathrm{d}\theta\,. \tag{2.41}$$

Using the result of equation (2.39) and due to the conditions imposed on the functions, the mass-loss formula follows

$$m_u = -\frac{1}{2}\int_0^\pi c_u^2 \sin\theta\,\mathrm{d}\theta\,. \tag{2.42}$$

We see that changes to the system can only do so through c_u, which therefore contains all the news in the system and receives the name of *news function*. Thus we reach the main statement in the publication of 1962 [4].

The mass of a system is constant if and only if there is no news. If there is news, the mass decreases monotonically as long as the news continues.

2.6 Chapter summary

In this chapter we wanted to describe the changes in the mass of a system due to the transmission of gravitational waves and the possible consequences of the non-linear equations. We started in Section 2.1 with a non-static, non-stationary, axially symmetric system considered at null infinity. We constructed, following the path originally taken by Bondi, a series of requirements that the metric describing such a system must fulfill. From these we reached the ansatz for the metric, as first proposed in the publication of 1960 [3]. Specifically, a metric characterized by a family of null hypersurfaces described by a retarded time, time-like coordinate, with an areal coordinate and null rays enumerated by a coordinate pair over a generalized sphere.

While stating Einstein vacuum equation in Section 2.2 we noted that three components of the Ricci tensor vanished identically. Moreover, we showed that the component R_{01} had also to vanish for the equation to hold. Thus, the original ten equations were reduced to only the four main equations (2.15), together with the supplementary conditions (2.21) and (2.22). These latter connect the integration functions of the main equations and restrain their behavior instead of introducing further free parameters.

In Section 2.3, through comparison of the Bondi metric with the Weyl metric (2.27) for a stationary interval of the system, we deduced an interpretation for the coefficients of the metric. In particular, we noted that according to the outgoing radiation condition γ can be expanded in a series of negative powers of r. As detailed in Appendix A, we considered the asymptotic behavior for $r \to \infty$ of the main equations. From plugging the inverse power ansatz in them, we derived also the coefficients for the expansions of β, U and V up to integration functions c, M, N, L and H.

Using again the asymptotic, $r \to \infty$, behavior of the main equations and the requirement of the metric $g_{00} = -1 + \mathcal{O}\left(r^{-1}\right)$, we concluded in Section 2.4 that $L = 0$. Thus, eliminating one of the integration functions. Likewise, we showed that H could be eliminated by an appropriate coordinate transformation. As briefly mentioned, the family of such transformations leaving the character of the metric unchanged constitute members of the known BMS (Bondi-Metzner-Sachs) group.

Finally, we closed the chapter in Section 2.5 by deriving the mass-loss and momentum-loss formulae from the supplementary conditions (2.21) and (2.22). For this, we used the fact that these conditions imply functions of the form $f(u, \theta) r^{-2}$. Thus, heavily simplifying calculations of the otherwise extremely complex components of the Ricci tensor. The results connect the mass aspect M and momentum aspect N to c, yielding the mentioned mass-loss and momentum-loss formulae. The u-derivative c_u, due to its role in controlling the flow of changes to the system, received the name of news function. In total, the main result of this chapter is that if γ, M and N are known at a specific instance in null-time $u = a$, and the news function c_u is known for all u in $a < u < b$ then the system is fully determined in the interval $a \leq u \leq b$ [4].

Chapter 3

The Bondi-Sachs metric and the memory effect

Chapter 2 was dedicated to the derivation of the Bondi metric, its solution of the Einstein vacuum equation and the consequential mass-loss and momentum-loss formulae, which explains outward going radiation as a consequence of the non-linear character of the equations. We considered the original results of Bondi, Van der Burg and Metzner. Accordingly, one of our starting assumptions was axial symmetry. In this chapter, however, we intent to drop this restriction by considering the, more general, Bondi-Sachs metric as introduced by Sachs in his paper of 1962 [12].

Section 3.1 motivates the Bondi-Sachs metric by construction from a scalar field enumerating light-like hypersurfaces. Also, the fall-off character of the metric and the series expansion of the coefficients are presented. In Section 3.2, we find, mostly by inspection, the values of the coefficients needed to derive the Bondi metric as special case of the Bondi-Sachs metric. Section 3.3 further continues the comparison between the Bondi and Bondi-Sachs metrics by stating the equations of motion, which are equivalent to the original main equations (2.15). We also present the mass-loss and momentum-loss formulae written in the generalized context of the Bondi-Sachs metric.

Rewriting the results of Chapter 2 in this new language has the advantage of allowing more abstract mathematical constructions to be built on top. In particular, using the concepts of symmetries and the algebras derived from them [7, 5]. This point is only briefly touched upon in Section 3.4, where we introduce the concept of the memory effect, which is closely related to this chapter's version of the news function, the news tensor.

3.1 The Bondi-Sachs metric

Let us consider a pseudo-Riemannian manifold, where there exists a scalar field u satisfying the equation $g^{ab}(\partial_a u)(\partial_b u) = 0$. Then analog to the argument in the previous chapter, the hypersurfaces $u =$ constant are light-like with their normal vector k^a, obeying the equations

$$k_a = \partial_a u, \qquad k^a k_a = 0, \qquad k^b \nabla_b k^a = 0. \tag{3.1}$$

A family of lines, called rays, can be generated by defining a line through each point as tangent to the vectors k^a at said point. If we further take a scalar field r to be a luminosity distance such that

$$r^4 \sin^2\theta = \text{constant}, \tag{3.2}$$

with coordinates θ and ϕ constant along each ray, it can be proved, that for every such manifold, the coordinates thus described obey this requirements only and only if the metric has the form [12, 13]

$$g = -2\mathrm{e}^{2\beta}\mathrm{d}u\,(\mathrm{d}r\; + F\mathrm{d}u\,) + r^2 q_{AB}(\mathrm{d}\sigma^A\; - U^A\mathrm{d}u\,)(\mathrm{d}\sigma^B\; - U^B\mathrm{d}u\,), \tag{3.3}$$

where σ_A, $A, B = 2, 3$ encapsulate the two angular coordinates and q_{AB} represents a two dimensional metric of said sub-manifold. Conversely, the inverse is given by

$$g^{-1} = 2F\mathrm{e}^{-2\beta}\partial_r\partial_r - 2\mathrm{e}^{2\beta}\partial_u\partial_r - 2\mathrm{e}^{-2\beta}U^A\partial_r\partial_A + r^{-2}q^{AB}\partial_A\partial_B. \tag{3.4}$$

It is relevant to explicitly mention that in this chapter we have, as seen in the metric (3.3), dropped the requirement for the system to be axially symmetric. We do however still consider a system which is asymptotically flat. Generally β, F, U^A and q_{AB} are functions of the coordinates (u, r, σ^A). The Bondi-Sachs metric is characterized by the following conditions

$$g_{11} = 0, \qquad\qquad g_{1A} = 0, \qquad\qquad (\sqrt{\det q})_r = 0. \tag{3.5}$$

and we still require the fall-off conditions [13]

$$g_{01} = -1 + \mathcal{O}\left(r^{-2}\right), \quad g_{0A} = \mathcal{O}\left(1\right), \quad g_{00} = \mathcal{O}\left(1\right), \quad g_{AB} = \mathcal{O}\left(1\right). \tag{3.6}$$

With this in mind, we introduce the following expansion for the metric coefficients, with an explicit dependence on r, so that all coefficients depend only on u and σ^A, see for example [5],

$$F(u, r, \sigma^A) = \bar{F}(u, \sigma^A) - \frac{M}{r} + \ldots, \tag{3.7}$$

$$\beta(u, r, \sigma^A) = \frac{\bar{\beta}(u, \sigma_A)}{r^2} + \ldots, \tag{3.8}$$

$$q_{AB}(u, r, \sigma^A) = \bar{q}_{AB}(u, \sigma^A) + \frac{c_{AB}}{r} + \ldots, \tag{3.9}$$

$$U^A(u, r, \sigma^A) = \frac{\bar{U}(u, \sigma^A)}{r^2} - \frac{2}{3r^3}\bar{q}^{AB}\left(\bar{P}^A + c_{BC}\bar{U}^C + \partial_B\bar{\beta}\right) + \ldots, \tag{3.10}$$

By considering the Riemann curvature tensor as $r \to \infty$, we can see that $\bar{F} = \frac{1}{2}$ is a requirement for all its components to vanish and is hence necessary for the metric to be asymptotically flat. Also, the derivative condition (3.5) implies that c_{AB} is a traceless tensor [7].

3.2 The Bondi metric as a special case

As it can be expected from the use of similar arguments in the choice of coordinates, the Bondi metric given in the previous chapter by equation (2.11) is in fact a special case of the Bondi-Sachs metric (3.3). To see this, we take again the angular coordinates $\sigma^A = (\theta, \phi)$[1] and define the components as follows:

$$
\begin{aligned}
F &= \frac{V_{\mathrm{Bondi}}}{2r}, \\
\beta &= \beta_{\mathrm{Bondi}}, \\
U^A &= (U_{\mathrm{Bondi}}, 0), \\
q_{AB} &= \begin{pmatrix} \mathrm{e}^{2\gamma} & 0 \\ 0 & \mathrm{e}^{-2\gamma}\sin^2\theta \end{pmatrix}, \\
\gamma &= \gamma_{\mathrm{Bondi}}.
\end{aligned} \tag{3.11}
$$

The statement $U^3 = 0$ is a reformulation of the requirement for the azimuth angle in the construction of the original Bondi metric. Following equation (3.9), q_{AB} is expanded in a series of inverse powers of r

$$
q_{AB} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} + \frac{1}{r}\begin{pmatrix} 2c & 0 \\ 0 & -2c\sin^2\theta \end{pmatrix} + \dots . \tag{3.12}
$$

3.3 The equations of motion and the news tensor

Solving Einstein vacuum field equations yields results analogous to the main equations of the Bondi metric (2.15). In fact, they are but a generalization of them, as it can be shown by substituting the values of the previous section. Solving the field equations order by order in powers of r yields for the first terms:

$$
(\bar{q}_{AB})_u = 0, \tag{3.13}
$$

$$
\bar{\beta} + \frac{1}{32} c_{AB} c^{AB} = 0, \tag{3.14}
$$

$$
\bar{R} - 4\bar{F} = 0, \tag{3.15}
$$

$$
\bar{U}^A + \frac{1}{2}\bar{\nabla}_B C^{AB} = 0, \tag{3.16}
$$

where $\bar{R}$ is the Ricci scalar and $\bar{\nabla}$ is the Levi-Civita connection associated with the metric $\bar{q}_{AB}$. As we previously mentioned, $\bar{F} = \frac{1}{2}$ must hold for the Bondi-Sachs metric to be asymptotically flat. Therefore, equations (3.13) and (3.15) imply that $\bar{q}_{AB}$ is the metric of a 2-dimensional manifold of constant positive curvature. Moreover, because q_{AB} is the metric of a sub-manifold of the Bondi-Sachs metric, excluding the time-like coordinate, $\bar{q}_{AB}$ has to be the metric of a Riemannian, compact and simply connected manifold, which, by the Jellett-Liebmann theorem [8, 10], is a round 2-sphere.

From the c_{AB} tensor we define the *news tensor* as follows

$$
N_{AB} = (c_{AB})_u. \tag{3.17}
$$

[1]Use of $q_{\theta\phi}$ and U^θ or U^ϕ would be sensible at this point, we avoid it, however, as to preserve consistency in the notation and prevent confusion with possible partial derivatives with respect to a coordinate.

From the Bondi-Sachs equivalent of the supplementary conditions we derive the equations describing the development of the system [7].

$$M_u = -\frac{1}{8}N_{AB}N^{AB} + \frac{1}{4}\bar{\nabla}_A\bar{\nabla}_B N^{AB}, \tag{3.18}$$

$$\begin{aligned}(\bar{P}_A)_u = \bar{\nabla}_A M + \frac{1}{8}\bar{\nabla}_A\left(c^{BC}N_{CB}\right) + \frac{1}{4}\bar{\nabla}_C\left(\bar{\nabla}_A\bar{\nabla}_B C^{BC} - \bar{\nabla}^C\bar{\nabla}^B C_{AB}\right)\\ + \frac{1}{4}\bar{\nabla}_B\left(N^{BC}c_{AC} - c^{BC}N_{AC}\right) - \frac{1}{4}N^{BC}\bar{\nabla}_A c_{BC}.\end{aligned} \tag{3.19}$$

Equation (3.18) is called the Bondi mass-loss formula. Note again, these are the generalizations of equations (2.39) and (2.40) from the previous chapter. For the first one, this can be seen directly by plugging in the expansion (3.11). The equation for the momentum aspect is however more involved. Suppose first that equations (2.40) and (3.19) are in some way connected and, thus, that both statements about the dynamic of the system are equivalent. Then, by equation (3.11), $U^A = (U_{\text{Bondi}}, 0)$. In particular, comparing the second terms of their respective series expansions, we have

$$2N + 3cc_\theta + 4c^2\cot\theta = -\frac{2}{3}\bar{q}^{2B}\left(\bar{P}_B + C_{BC}\bar{U}^C + \partial_B\bar{\beta}\right). \tag{3.20}$$

Solving for N yields

$$N = -\frac{2c^2\cot\theta + 2cc_\theta + \bar{P}^2}{3}. \tag{3.21}$$

Next, we differentiate with respect to u and plug in $(\bar{P}_2)_u$ from equation (3.19), which results in (2.40) identically. Hence both equations for the momentum aspect are equivalent if and only if we define N by equation (3.20) and equation (3.19) holds.

3.4 The memory effect

In this last section we intent to argue for the possibility of memory effects in the context of gravitational waves in general relativity. By this we mean permanent relative displacement between masses due to a burst of gravitational waves passing through. The presented argument follows closely the one in [15, 5].

Let us consider the deviation equation for particles in free fall:

$$(v^a\nabla_a)^2\xi^b = -R_{acd}{}^b v^a v^d \xi^c \tag{3.22}$$

We assume to be near null-infinity, such that $v^a = \delta^a_0$ and equation (3.22) becomes

$$(\xi^\mu)_{uu} = -R_{0\alpha 0}{}^\mu \xi^\alpha. \tag{3.23}$$

Furthermore, because $r \to \infty$ we can replace the Riemann tensor to first order by the Weyl tensor

$$C_{abcd} = R_{abcd} - \left(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\right) + \frac{1}{3}\left(R\,g_{a[c}g_{d]b}\right). \tag{3.24}$$

Additionally, since for the initial value ξ_0^α holds $\xi^\alpha - \xi_0^\alpha = \mathcal{O}\left(r^{-1}\right)$, we have

$$\begin{aligned} \xi^\mu|_{u=-\infty}^{u=+\infty} &= -\int_{-\infty}^{\infty} \mathrm{d}u' \int_{-\infty}^{\infty} \mathrm{d}u \; C_{0\alpha 0}{}^{\mu} \xi_{(0)}^{\alpha} \\ &= \Delta^\mu{}_\alpha \xi_0^\alpha, \end{aligned} \tag{3.25}$$

where the leading order contribution of the Weyl tensor can be expressed through the news tensor as follows:

$$C_{0A0B} = -\frac{1}{2}(N_{AB})_u. \tag{3.26}$$

We observe how the existence and evolution of news corresponds to a deviation of the worldline of a particle and thus to a permanent displacement due gravitational waves.

3.5 Chapter summary

In Section 3.1, we introduced the Bondi-Sachs metric by taking a coordinate u to enumerate a family of light-like hypersurfaces. Moreover, we chose a further r coordinate fulfilling equation (3.2) and two angle-like coordinates σ^A, which are constant along rays generated by the normal vectors of the surfaces. Here, in contrast to the Bondi metric, we dropped the requirement of axial symmetry. Section 3.2 presented a way to translate the metric from Chapter 2 into this new form. We then used in Section 3.3 an ansatz for the parameters of the Bondi-Sachs metric based on its fall-off characteristics. Einstein vacuum field equations provided the relations determining the terms of the ansatz in a manner analog to the main equations (2.15). They also yielded the dynamic constraints in the form of the mass-loss and momentum-loss formulae. Lastly, in Section 3.4 we considered the memory effect, a permanent relative displacement due to the propagation of gravitational waves and described by the flow of news through a region of spacetime.

Chapter 4

Further developments

Throughout this thesis we recollected the fundamental results of the Bondi-Sachs formalism. In Chapter 2, we constructed the original Bondi metric (2.11) in detail, and in Chapter 3 we introduced the Bondi-Sachs metric (3.3), proving that it corresponds to a generalized Bondi metric. As explained in the introduction, the intention behind this is to look into the possibility of extending the Bondi-Sachs formalism to more general alternative theories of gravity. In the literature, there exists multiple families of such theories. We point the reader to [14] for a list of some examples.

We speculate that the differences in mass-loss and momentum-loss formulae, and the memory effect could lead to an observable to distinguish various modified gravity theories. A fingerprint of sorts.

As a more concrete example, also of the possible difficulties that may arise in the process, we consider a scalar-tensor theory, derived from the action of a lagrangian density like [16]

$$\mathcal{L} = \sqrt{-\det g}\,[h(\phi)R + l(\phi)\partial^\mu\phi\partial_\mu\phi + 2\lambda(\phi)] + \mathcal{L}[\psi^2(\phi)g_{\mu\nu}, \dots], \tag{4.1}$$

where ϕ is the scalar coupled to the gravitational field, R is the Ricci scalar of the metric, h, l and λ are arbitrary functions.

Let us assume the supplementary conditions are known for some scalar-tensor, or some other alternative theory. Having extra degrees of freedom creates the expectation to find further channels through which energy is radiated away from the system under consideration. Hence a different memory effect is to be expected. Therefore, as explained, the idea of this proposal is to use this effect as a fingerprint for discarding or validating such alternative theories.

Notably, however, the Bondi-Sachs formalism in the classical theory relies heavily on the result $R_{\mu\nu} = 0$ to reduce the ten-equations system into only four together with two supplementary conditions. This is no longer possible in a scalar-tensor theory, for which, in general, $R_{\mu\nu} \neq 0$. Therefore, it may be necessary to develop new methods for such a generalization.

Conclusion

In this thesis, we considered asymptotically flat metrics based upon coordinates describing light-like hypersurfaces with normals along which energy is radiated. Our initial construction in Chapter 2 used axial symmetry and resulted in the Bondi metric (2.11). In Sections 2.2 and 2.3 we solved the gravitational vacuum field equations, which were simplified by the form of the metric. The result were the equations for the coefficients of the metric (2.35) to (2.38) and the mass-loss (2.39) and momentum-loss (2.40) formulae, which describe the system's outwards flow of energy as a consequence of the non-linear nature of Einstein field equations. While deriving the results of Chapter 2, we briefly touched upon the topic of the asymptotic isometries of the Bondi metric. These constitute an infinite group in addition to the known Poincaré group and have altogether the name of BMS group. As mentioned, there exists recent interest to use this group together with whole of the Bondi-Sachs formalism as a possible guide in the search for quantum gravity [5, 7].

The axial symmetry requirement of the initial construction was later dropped in Chapter 3, where we introduced the more general Bondi-Sachs metric (3.3). We briefly discussed in Section 3.2 how to translate the previous metric into this new form. Solving Einstein vacuum field equations for the Bondi-Sachs metric, order by order in powers r, we reached in Section 3.3 a result equivalent to the main equations (2.15) from Chapter 2. Through the definition of the news tensor N_{AB}, the equivalent of the news function c_u, we also derived from the field equations the mass-loss (3.18) and momentum-loss (3.18) formulae in the context of the Bondi-Sachs metric. Finally, in Section 3.4, we showed how the news tensor (3.17) is related to the memory effect, a permanent relative displacement due to the propagation of gravitational waves and described by the flow of news through a region of spacetime.

We close this thesis in Chapter 4, where we suggested to use the results of the loss formulae and memory effect as a way that could possibly lead to an observable to distinguish various modified gravity theories. The future objective of the author is to further expand this formalism for this very same purpose.

Another interesting question, though in this work left unanswered, is the meaning of mass in the context of this formalism. The result we proved to change as a consequence of radiation of energy receives the name of Bondi mass. As such, it is, strictly speaking, the mass of the system that changes. In a sense, it is a consequence of the definition, rather than being necessarily a consequence of deeper physical nature. There exists the connection with the mass term in the Weyl metric, but how is this related to other notions of mass in, for example, flat spacetime? Leaving the known energy mass relation aside, what changes when two masses orbit each other? Is there more than the potential energy lost through orbital decay? If the particles considered have fixed, unchangeable mass, like the fundamental particles of the standard model. Can we still expect the release of gravitational waves from their interaction? The original work of Bondi, Van der Burg and Metzner does describe the possibility of non-radiative motions. However, this leaves the question open of how this restricts the degrees of freedom of such a system.

Appendix A

Asymptotic character of the main equations

This Appendix complements Section 2.3 of Chapter 2. We consider in further detail the fall-off character of the main equations and its consequences for the components of the Bondi metric (2.11). We only make arguments for the order of the functions and their boundness with respect to $r \to \infty$. The actual coefficients of the series expansions are, as described in Chapter 2, recovered by plugging known expansions into the equations.

A.1 First main equation

Equation (2.23) is given by

$$-4r^{-1}[\beta_r - \frac{1}{2}r\gamma_r^2] = 0,$$

with the following fall-off characteristics

$$\gamma_r = \mathcal{O}\left(r^{-2}\right).$$

Thus, as $r \to \infty$

$$\begin{aligned} &\beta_r = \frac{1}{2}r\gamma_r^2 = \mathcal{O}\left(r^{-3}\right) \\ &\Rightarrow \beta = H(u,\theta) + \mathcal{O}\left(r^{-2}\right). \end{aligned}$$

A.2 Second main equation

Equation (2.24) is given by

$$[r^4 \mathrm{e}^{2(\gamma-\beta)}U_r]_r - 2r^2[\beta_{r\theta} - \gamma_{r\theta} + 2\gamma_r\gamma_\theta - 2r^{-1}\beta_\theta - 2\gamma_r \cot\theta] = 0,$$

with the following fall-off characteristics:

$$\begin{aligned} \beta_{r\theta} &= \mathcal{O}\left(r^{-3}\right), & \gamma_{r\theta} &= \mathcal{O}\left(r^{-2}\right), & \gamma_r\gamma_\theta &= \mathcal{O}\left(r^{-3}\right), \\ 2r^{-1}\beta_\theta &= 2r^{-1}H_\theta + \mathcal{O}\left(r^{-3}\right), & 2\gamma_r\cot\theta &= \mathcal{O}\left(r^{-2}\right). \end{aligned}$$

Thus, as $r \to \infty$

$$
\begin{aligned}
&[r^4 \mathrm{e}^{2(\gamma-\beta)} U_r]_r + 4rH_\theta + K = 0 \\
&\Rightarrow r^4 \mathrm{e}^{2(\gamma-\beta)} U_r + 2r^2 H_\theta + Kr - 6N = 0 \\
&\Rightarrow U_r = \mathrm{e}^{2H} \left(6Nr^{-4} - 2H_\theta r^{-2} - Kr^{-3} \right) \\
&\Rightarrow U = \mathrm{e}^{2H} \left(2H_\theta r^{-1} + K/2r^{-3} - 2Nr^{-3} \right) + L.
\end{aligned}
$$

Hence

$$
U = L(u, \theta) + \mathcal{O}\left(r^{-1}\right).
$$

A.3 Third main equation

Equation (2.25) is given by

$$
\begin{aligned}
2V_r &+ \frac{1}{2} r^4 \mathrm{e}^{2(\gamma-\beta)} U_r^2 - r^2 U_{r\theta} - 4rU_\theta - r^2 U_r \cot\theta - 4rU \cot\theta \\
&+ 2\mathrm{e}^{2(\beta-\gamma)} [-1 - (3\gamma_\theta - \beta_\theta)\cot\theta - \gamma_{\theta\theta} + \beta_{\theta\theta} + \beta_\theta^2 + 2\gamma_\theta(\gamma_\theta - \beta_\theta)] = 0,
\end{aligned}
$$

with the following fall-off characteristics:

$$
\begin{array}{ccc}
r^4 U_r^2 = \mathcal{O}(1), & r^2 U_{r\theta} = \mathcal{O}(1), & rU_\theta = rL_\theta + \mathcal{O}(1), \\
r^2 U_r \cot\theta = \mathcal{O}(1), & rU\cot\theta = rL + \mathcal{O}(1), & (3\gamma_\theta - \beta_\theta)\cot\theta = -H_\theta + \mathcal{O}\left(r^{-1}\right), \\
\gamma_{\theta\theta} = \mathcal{O}\left(r^{-1}\right), & \beta_{\theta\theta} = H_{\theta\theta} + \mathcal{O}\left(r^{-2}\right), & \gamma(\gamma_\theta - \beta_\theta) = \mathcal{O}\left(r^{-1}\right).
\end{array}
$$

Thus, as $r \to \infty$

$$
\begin{aligned}
&2V_r + K_1 \mathrm{e}^{2H} - 4rL_\theta - 4rL + K_2 + 2\mathrm{e}^{2H}[-1 + H_\theta \cot\theta + H_{\theta\theta} + H_\theta^2] = 0 \\
&\Rightarrow 2V - 4M + K_1 r \mathrm{e}^{2H} - 2r^2 L_\theta - 2r^2 L + K_2 r \\
&\qquad + 2r\mathrm{e}^{2H}[-1 + H_\theta \cot\theta + H_{\theta\theta} + H_\theta^2] = 0 \\
&\Rightarrow V = 2(L + L_\theta) r^2 + \mathcal{O}(r) = \mathcal{O}\left(r^2\right).
\end{aligned}
$$

Taking into account the condition of g_{00} to remain bounded and thus $L = 0$, it follows

$$
V = \mathcal{O}(r).
$$

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