Conformal and disformal transformations using Mathematica

A case study in xAct

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- Code available to download (see last slide)

Conformal transformation

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(Based on Appendix D of Wald 1984)

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$$h_{ab} = \Omega^2 g_{ab}$$
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where

$$C^c_{ab} = 2\delta^c{}_{(a} \nabla_{b)} \ln \Omega - g_{ab} g^{cd} \nabla_d \ln \Omega$$

Conformal transformation

Finally, the curvature $\tilde{R}_{abc}{}^{d}$, associated with D_a and the curvature, $R_{abc}{}^{d}$, associated with ∇_a is given by:

$$\begin{aligned}
\tilde{R}_{abc}{}^{d} = R_{abc}{}^{d} &+ 2\delta^{d}{}_{[a}\nabla_{b]}\nabla_{c}\ln\Omega \\
&- 2g^{de}g_{c[a}\nabla_{b]}\nabla_{e}\ln\Omega \\
&+ 2\left(\nabla_{[a}\ln\Omega\right)\delta^{d}{}_{b]}\nabla_{c}\ln\Omega \\
&- 2\left(\nabla_{[a}\ln\Omega\right)g_{b]c}g^{df}\nabla_{f}\ln\Omega \\
&- 2g_{c[a}\delta^{d}{}_{b]}g^{ef}\left(\nabla_{e}\ln\Omega\right)\nabla_{f}\ln\Omega
\end{aligned}$$

Disformal transformation

Disformal transformation

(Based on Alinea 2020)

We introduce a new metric h_{ab} defined by:

$$h_{ab} = A(\phi)g_{ab} + B(\phi)(\nabla_a\phi)\nabla_b\phi$$
$$h^{ab} = \frac{1}{A}\left(g^{ab} - \frac{B}{A-2BX}(\nabla^\mu\phi)\nabla^\nu\phi\right)$$

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where the inverse can be derived using the *Sherman-Morrison formula*:

$$(\mathbf{M} + \mathbf{u}\mathbf{v}^T)^{-1} = \mathbf{M}^{-1} - \frac{\mathbf{M}^{-1}\mathbf{u}\mathbf{v}^T\mathbf{M}^{-1}}{1 + \mathbf{v}^T\mathbf{M}^{-1}\mathbf{u}}$$

Disformal transformation

The transformed Christoffel symbols are given by:

$$
\begin{aligned}
\hat{\Gamma}^{\alpha}_{\mu\nu} = & \quad \Gamma^{\alpha}_{\mu\nu} + C^{\alpha}_{\mu\nu} \\
= & \quad \Gamma^{\alpha}_{\mu\nu} + \frac{A'}{2A}\left(\delta^{\alpha}_{\mu}\nabla_{\nu}\phi + \delta^{\alpha}_{\nu}\nabla_{\mu}\phi\right) - \frac{A'}{2(A-2BX)}g_{\mu\nu}\nabla^{\alpha}\phi \\
& \quad + \frac{AB' - 2A'B}{2A(A-2BX)}(\nabla^{\alpha}\phi)(\nabla_{\mu}\phi)\nabla_{\nu}\phi \\
& \quad + \frac{B}{A-2BX}(\nabla^{\alpha}\phi)\nabla_{\mu}\nabla_{\nu}\phi
\end{aligned}
$$

where

$$X = -\frac{1}{2}g^{\mu\nu}(\nabla_{\mu}\phi)\nabla_{\nu}\phi$$

Disformal transformation

Finally, the transformed Riemann curvature tensor is given by:

$$
\begin{aligned}
\hat{R}^{\alpha}{}_{\mu\beta\nu} = & \quad -\partial_\nu \hat{\Gamma}^{\alpha}_{\mu\beta} + \partial_\beta \hat{\Gamma}^{\alpha}_{\mu\nu} - \hat{\Gamma}^{\rho}_{\mu\beta}\hat{\Gamma}^{\alpha}_{\rho\nu} + \hat{\Gamma}^{\rho}_{\mu\nu}\hat{\Gamma}^{\alpha}_{\rho\beta} \\
= & \quad R^{\alpha}{}_{\mu\beta\nu} - \partial_\nu C^{\alpha}_{\mu\beta} + \partial_\beta C^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\rho\nu} C^{\rho}_{\mu\beta} - \Gamma^{\rho}_{\mu\beta} C^{\alpha}_{\rho\nu} \\
& \quad C^{\rho}_{\mu\beta} C^{\alpha}_{\rho\nu} + \Gamma^{\alpha}_{\rho\beta} C^{\rho}_{\mu\nu} + \Gamma^{\rho}_{\mu\nu} C^{\alpha}_{\rho\beta} + C^{\rho}_{\mu\nu} C^{\alpha}_{\rho\beta}
\end{aligned}
$$

Sources

Alinea, Allan L. (2020). *On the Disformal Transformation of the Einstein-Hilbert Action*. arXiv: 2010.00956 [gr-qc].

Martín-García, José M. (2025). *xAct: Efficient tensor computer algebra for the Wolfram Language*. URL: https://josmar493.dreamhosters.com/.

Wald, R.M. (1984). *General Relativity*. University of Chicago Press. ISBN: 9780226870373.

Mathematica notebooks and these slides are available on my website

https://thomaslabs.org/research/xact-example.html